G027

Sampling Functions and Sparse Reconstruction Methods

M. Naghizadeh* (University of Alberta) & M.D. Sacchi (University of Alberta)

SUMMARY

In this paper we investigate the effects of different sampling operators on the performance of sparse reconstruction methods. The common paradigm in seismic data processing is to favor regular sampling. We will show, however, that regular sampling often hampers our data recovery efforts. Random sampling, on the other hand, can lead to algorithms where the reconstruction is almost perfect when the underlying spectrum of the signal can be assumed sparse. Also, simple 1D, 2D and 3D synthetic examples are provided to test the sparse reconstruction of signals sampled by various sampling functions.

Introduction

The sampling theorem is an interesting topic of chief importance in the physical sciences and engineering. Sampling methods can be classified into different groups. The most important distinction is between uniform and nonuniform sampling methods. In the uniform sampling method, a signal is sampled periodically at constant intervals. This leads to recovery conditions based on the well-known Nyquist sampling theorem. An overview of uniform sampling and its properties is given by Unser (2000). Meanwhile, in the nonuniform sampling case samples are picked in an irregular fashion. Belmont (1993) gives a nice explanation of nonuniform sampling operators.

Regardless of the adopted sampling method, the truly important step is to reconstruct the original signal from sampled data. Reconstruction algorithms utilize specific assumption about the original signal. The most commonly used assumption is the band-limitation. In other words, band-limited signals can completely be recovered from a wide range of sampling methods. Duijndam et al. (1999) utilized band-limiting assumptions in order to reconstruct the irregularly sampled seismic records in the spatial directions. Another important assumption is the sparseness of a signal in the Fourier domain. Liu and Sacchi (2004) used iteratively updated weighting functions of the data spectrum as a sparsity constraint. Zwartjes and Gisolf (2006) compared the performance of several sparsity norms on seismic data and simply addressed the methods as the Fourier reconstruction. Naghizadeh and Sacchi (2007) used band-limiting operators to reconstruct the low frequency portion of the aliased seismic data. Subsequently, they used Multi-Step Auto-Regressive (MSAR) operator to calculate prediction filters to reconstruct all the frequencies.

In this paper we will investigate the effects of various sampling schemes on the performance of sparse reconstruction algorithms. First, by developing a mathematical framework for general sampling operators, we try to understand the differences between uniform and nonuniform sampling operators and their spectra. Later, using simple 1D examples we will show the reconstruction results for different sampling operators. Also, we will provide examples of 2D and 3D sparse data reconstruction with different sampling operators.

Sampling function

Consider a length-N discrete signal $\mathbf{x} = \{x(0), x(1), x(2), \ldots, x(N-1)\}$. Further we assume that only M samples of the signal $\mathbf{x}$ with the locations $\mathbf{h} = \{h(0), h(1), h(2), \ldots, h(M-1)\}$ are available and the rest are set to zero. The signal with missing samples can be represented as:

$$\mathbf{x_s} = x_s(n) = \begin{cases} x(n) & n \in \mathbf{h} \\ 0 & n \notin \mathbf{h} \end{cases} \qquad (1)$$

The Discrete Fourier Transform (DFT) of $\mathbf{x}$ is obtained by:

$$\mathbf{X} = X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-i2\pi nk}{N}} \qquad k = 0,1,2,\ldots,N-1. \qquad (2)$$

Defining the scaling function $\mathbf{Q}$ as:

$$\mathbf{Q} = Q(k) = \frac{1}{N} \sum_{m=0}^{M-1} e^{\frac{-i2\pi h(m)k}{N}} \qquad k = 0,1,2,\ldots,N-1, \quad (3)$$

and using orthogonal properties of Fourier series, it can be easily proved that $\mathbf{X_s} = \mathbf{X} * \mathbf{Q}$, where $\mathbf{X_s}$ is the Fourier transform of signal with missing samples and $*$ represents the Convolution operator.

Analyzing the behavior of scaling function for different sampling scheme such as uniform, nonuniform, and regular sampling with gaps is the key to understand the nature of artifacts caused by each sampling functions. Figure 1a shows the value of the scaling function for the decimation factors of r = *1, 2, 3, 4* and *5* from top to bottom, respectively (r =*1* means no decimation (original signal); r =*2* means removing every other sample, r = *3* means keep 1 remove *2*). Figure 1b and 1c show the scaling function for 5 different realizations of nonuniform sampling while 50% and 90% of the samples are missing, respectively. The scaling function of nonuniform sampling operator has different structure than the uniform one. It has nonzero values for the all of its components inside the fundamental interval of the spectrum. Interestingly, the scaling function for nonuniform sampling, with 50% of samples missing, has only one high value appears at the exact location of the uniform sampling case with no decimation ($r = 1$). The rest of its components are small (Figure 1b). As the number of missing samples increases more components of the scaling function become bigger. This simple representation of the behavior of the scaling function for uniform and nonuniform sampling explains the very reason of the success and failure of the sparse reconstruction methods for each of these individual cases. With regularly missing samples, the spectrum of the sampled signal has artifacts as big as the original signals spectrum and any sparse reconstruction method would preserve these unwanted artifacts as if they were part of the original spectrum. On the other hand, for the case of nonuniform sampling (with a sufficient number of available samples) the only high amplitude repeat of the original spectrums is the main spectrum and as a result, sparse reconstruction has the potential to recover the original spectrum. Figure 1d shows the plot of scaling function for different size of gaps, which the size of gaps is decreasing from top to bottom. This shows that the gap inside data leads to repetition of the spectrum of signal in its neighbor samples. The amplitude of the repeated spectrum increases as the size of the gap increases. This means sparse reconstruction methods can, in fact, recover the gaps inside data if the size of the gap is not big.

Examples

In order to examine the performance of sparse reconstruction methods with various sampling operators, a synthetic 1D signal with two harmonics were created (Figure 2a). For 1D examples, left panels will show the spatial domain (or time domain) and right panels will represent their correspondent spectra. To proceed with the first example, 80 percent of data was eliminated randomly and the missing samples (Figure 2b) were reconstructed using sparseness constraints. Figure 2e shows the result of reconstruction. Since available samples were picked by a random nonuniform sampling operator the reconstruction was successful. Figures 2b, 2d, and 2f show the Fourier domain of Figures 2a, 2c, and 2e, respectively.

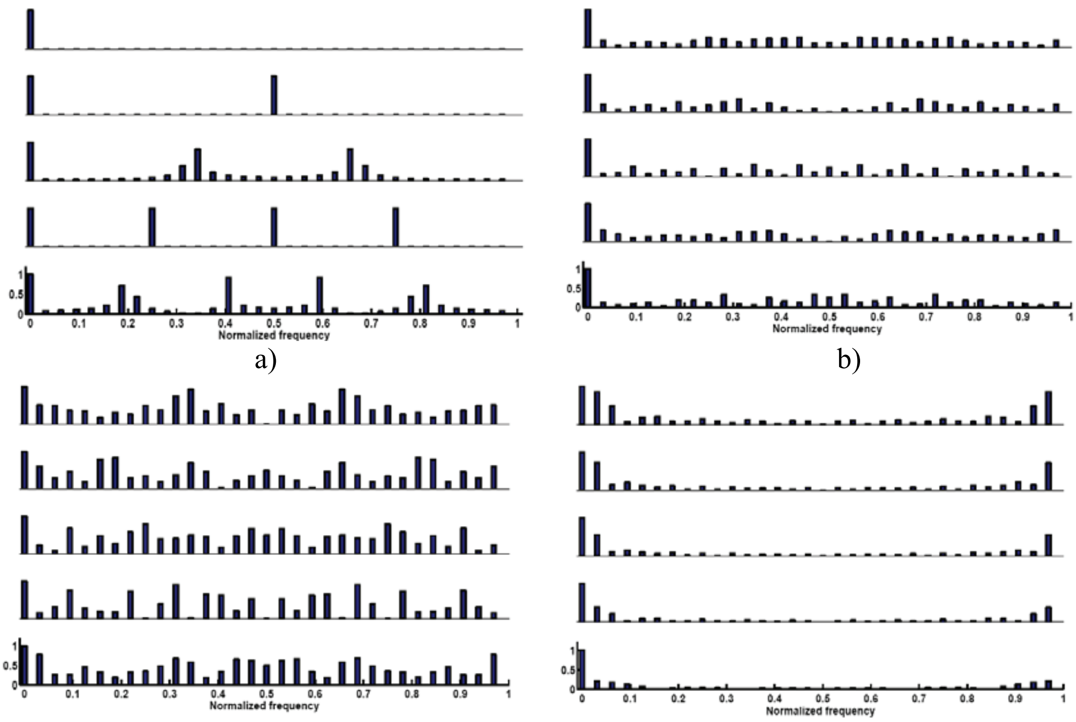


Figure1: Representations of scaling function for a) Regular decimation of signal; b,c) random sampling with 50% and 90% missing samples, respectively; d) gap inside the data

Figure 3a shows an example on which the available samples have been randomly picked from an already decimated original signal by a factor of 2. Since the sampling operator has picked the samples from an already decimated signal (which has two main spikes in the scaling function) sparse reconstruction was only able to reconstruct the decimated signal (Figure 3c). Figures 3b and 3d show the Fourier domain of Figures 3a and 3c, respectively.

Figure 4a shows a 2D synthetic example. A gap is made inside data by eliminating some of the traces (Figure 4b). Figure 4c shows the result of sparse reconstruction of the data with gaps.

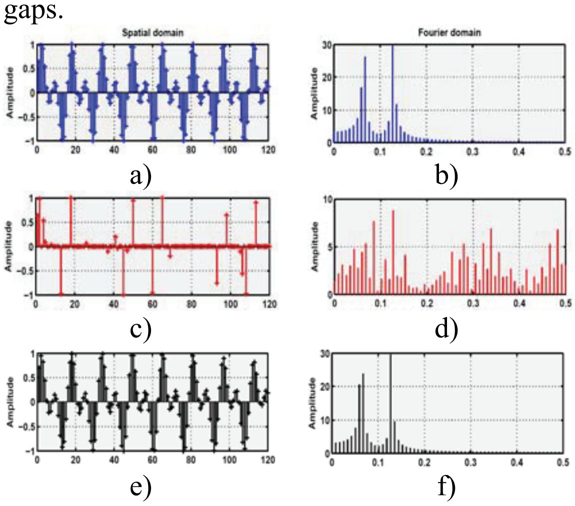


Figure2: a) Original signal; c) Signal with 80% randomly missing samples; e) Reconstructed signal using sparsity constraint. b, d, f) Fourier representations of a, c, e, respectively.

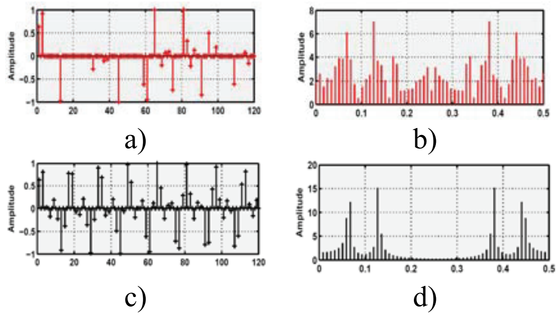


Figure3: a) Signal with 80% missing samples. Samples are picked from the decimated signal by factor of 2. c) Sparse reconstruction of a. b, d) Fourier representations of a, c, respectively.

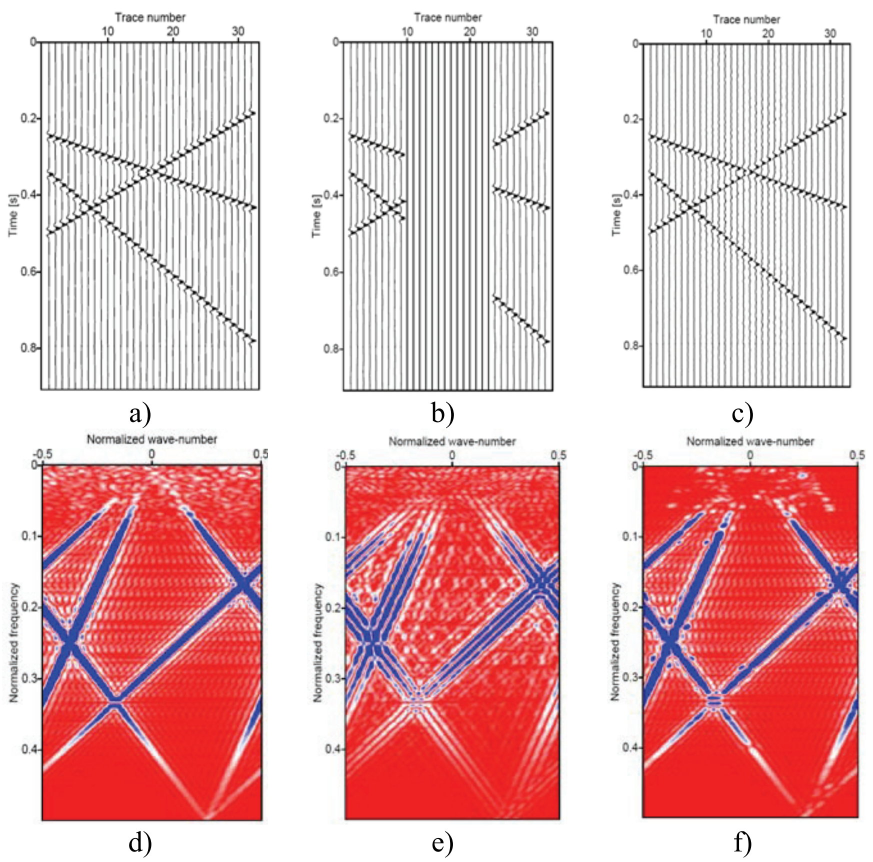
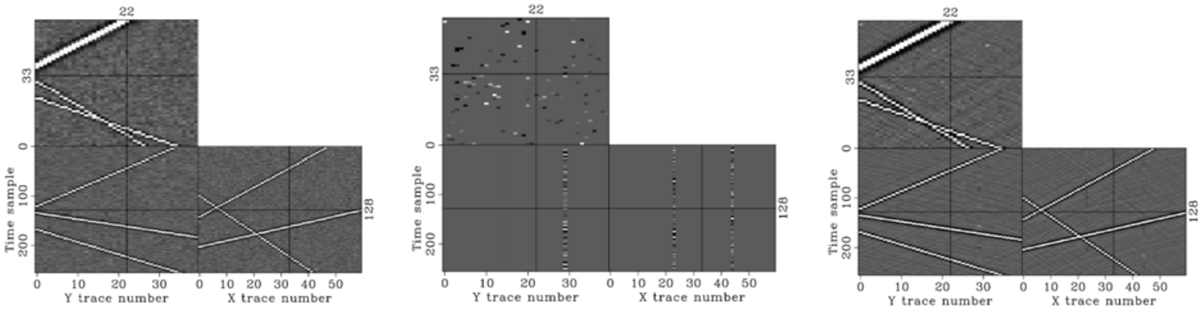


Figure4: a) Original 2D synthetic data; b) Data with gap; c) sparse reconstruction of b. d, e, f) Fourier representations of a, b, c, respectively.



a) b) c)

Figure5: a) Original 3D synthetic data b) 90% randomly missing data c) sparse reconstruction result

Figures 4d, 4e and 4f show the Fourier representations of Figures 4a, 4b and 4c, respectively. As it was expected from the shape of scaling function for the data with gaps, the gap causes low amplitude repetition of the spectrum in the vicinity of the original spectrum. It can be concluded that the sparse reconstruction methods are good candidates for reconstruction of gaps inside the data.

To examine the above mentioned performances for a 3D case, a synthetic cube of data with three plane events are created (Figure 5a). Further, 90 percent of the traces are randomly removed. The cube of randomly missing traces (Figure 5b) were reconstructed using sparse reconstruction algorithm, on which the sparseness constraint is imposed on each plane of frequencies to recover the frequency slices. As it was shown in Figure 5c, the reconstruction was successful, because the sampling had been done in a random fashion.

Conclusion

In this paper we examined the effects of random sampling and regular sampling operators on the performance of sparse Fourier reconstruction methods. The spectrum of the sampling operator is convolved with the spectrum of original data to give the spectrum of sampled data. Provided that the original data had a sparse spectrum, its convolution with the spectrum of the random sampling operator (which has only one dominant component) will create a signal with sparse Fourier spectrum with small amplitude artifacts. Therefore, randomly sampled signals are recoverable by imposing sparseness constrains in the Fourier domain. Conversely, since the spectrum of the regular sampling operator contains more than one strong component, its convolution with the spectrum of original data will contain unwanted repetitions of the spectrum of the original data. Hence, eliminating the artifacts caused by the regular sampling operator (which are as high in amplitude as the spectrum of the original data) using sparse reconstruction methods would be impossible. In the cases of data with gaps, the spectrum of sampling function causes small amplitude repetition of the original spectrum in the vicinity of the original spectrum. Therefore, sparse reconstruction algorithms are also suitable for the reconstruction of data with gaps.

References

Belmont, M. [1993] Nonuniform sampling specifically for finite-length data. IEE Proceedings, 140 (1), 55–62.

Duijndam, A., M. Schonewille, and C. Hindriks [1999] Reconstruction of band-limited signals, irregularly sampled along one spatial direction. Geophysics, 64 (2), 524–538.

Liu, B. and M. D. Sacchi [2004] Minimum weighted norm interpolation of seismic records. Geophysics, 69 (6), 1560–1568.

Naghizadeh, M. and M. D. Sacchi [2007] Multi-step auto-regressive reconstruction of seismic records. Geophysics, 72 (6), V111–V118.

Unser, M. [2000] Sampling - 50 years after shannon. Proc. of the IEEE, 88 (4), 569–587.

Zwartjes, P. and A. Gisolf [2006] Fourier reconstruction of marine-streamer data in four spatial coordinates. Geophysics, 71 (6), V171–V186.