FX ARMA Filters

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Summary

We present and develop an algorithm to retrieve harmonic signals immersed in white noise. The algorithm is applied in the $f-x$ domain to enhance the signal-to-noise-ratio of 2D seismic wavefields.

Classical $f-x$ noise attenuation techniques are based on autoregressive (AR) modeling. In this problem the noise is considered a sequence of random innovations rather than an additive process. We show that linear events immersed in additive white noise can be properly represented in the $f-x$ domain by means of an autoregressive/moving-average (ARMA) model. The ARMA structure of the signal leads, in the stationary approximation, to a eigenvalue problem. In fact, the prediction error filter (PEF) is obtained from the eigen-decomposition of the covariance matrix of the noisy signal. The PEF is applied to the noisy data and finally, an estimate of the additive noise sequence is obtained by self-deconvolving the PEF from the filtered data. We also examine the similarities of our algorithm to the projection filtering technique proposed by Soubaras (1994, 1995).

Introduction

In 1984 Canales showed how to design prediction error filters (PEF) in the $f-x$ domain to extract the predictable part of the signal from the data. This technique is optimal if the signal in the $f-x$ space can be modeled via an autoregressive (AR) model at each frequency f. In this presentation we show that an optimal model for linear events in the $f-x$ domain is given by an autoregressive/moving-average (ARMA) model. These type of models have been extensively studied in the context of harmonic retrieval and are associated to the well known Pisarenko harmonic spectral estimator (Pisarenko, 1973; Ulrych and Clayton, 1976). In this paper we utilize the ARMA structure of the signal to estimate the PEF, the noise sequence is estimated by self-deconvolving the PEF from the filtered data.

The signal model

We first consider a signal, $s(t,x)$, composed of a single waveform with constant ray parameter ψ. The frequency domain representation of $s(t,x)$ is given by

$$S(f,x) = A(f)\, e^{i2\pi f \psi x}\,, \qquad (1)$$

where $A(f)$ indicates the source spectrum, f the temporal frequency, and x the spatial variable or offset. We will assume that the spatial variable x is regularly discretized according to $x=(k-1)\Delta x$, $k=1:N$. For any temporal frequency, f, we can write

$$S_n = A\, e^{i\alpha n}, \qquad n=1,N \qquad (2)$$

where $\alpha = 2\pi f \psi \Delta x$. The signal can be easily predicted using a first order difference equation. The following recursion is obtained by combining S_n and S_{n-1}

$$S_n = a_1 S_{n-1}\,. \qquad (3)$$

where $a_1 = exp(i\alpha)$. Similarly, it can be shown that the superposition of p complex harmonics (p linear events in $x-t$) can be recursively represented by a difference equation of order p

$$S_n = a_1 S_{n-1} + a_2 S_{n-2} + \ldots a_p S_{n-p}\,. \qquad (4)$$

The latter can be written in prediction error form as follows

$$\sum_{k=0}^{p} g_k S_{n-k} = 0\,, \qquad (5)$$

$g_0 = 1$ and $g_k = -a_k, \quad k=1,p$.

Adding noise ,W_n, to the data gives rise to the following process

$$Y_n = S_n + W_n\,, \qquad (6)$$

substituting $S_{n-k} = Y_{n-k} - W_{n-k}$ into equation (5) yields

$$\sum_{k=0}^{p} g_k Y_{n-k} = \sum_{k=0}^{p} g_k W_{n-k} = e_n\,. \qquad (7)$$

The latter is an ARMA(p,p) process in which the AR and MA coefficients are identical. The signal e_n is used to designate the non-white innovation sequence $\sum_{k=0}^{p} g_k W_{n-k}$. Ulrych and Clayton (1976) discussed the relationship of the Pisarenko harmonic spectral estimator to this special type of ARMA model.

Equation (7) can be written in matrix form as follows:

$$\mathbf{Yg} = \mathbf{Wg} = \mathbf{e}\,, \qquad (8)$$

where $\mathbf{Y}$ is the convolution matrix of the signal with entries given by the noisy sequence Y_k properly shifted and padded with zeros in order to express discrete convolution. Similarly, $\mathbf{W}$ is the convolution matrix of the noise with entries given by the unknown noise sequence W_k. If we assume that the noise is a zero mean white process and that the signal and the noise are uncorrelated, the PEF $\mathbf{g}$ can be estimated by transforming equation (8) into the following eigen problem:

$$\mathbf{R}_Y \mathbf{g} = \sigma_W^2 \mathbf{g} \,. \tag{9}$$

The matrix $\mathbf{R}_Y$ is the correlation matrix of the noisy data, σ_W^2 is the variance of the noise. It is intuitively clear that the desired PEF is the eigenvector that corresponds to the minimum eigenvalue of $\mathbf{R}_Y$. The minimum eigenvalue is an estimate of the noise variance σ_W^2.

Noise estimation

After estimating $\mathbf{g}$ the remaining problem is to estimate the noise sequence $\hat{W}_k$ which will be used to estimate the "clean" signal $\hat{S}_k = Y_k - \hat{W}_k$. The noise is estimated by deconvolving the PEF from the non-white innovation in equation (8). In order to facilitate the algebra we will rewrite equation (8) by commuting the sequences involved in the convolution

$$\mathbf{G}\mathbf{y} = \mathbf{G}\mathbf{w} = \mathbf{e} \,. \tag{10}$$

$\mathbf{G}$ is now the convolution matrix of the PEF, $\mathbf{y}$ and $\mathbf{w}$ are vectors containing the observations and the white noise sequence, respectively. The noise sequence is estimated from the colored innovation $\mathbf{e} = \mathbf{G}\mathbf{w}$ by deconvolving the PEF,

$$\hat{\mathbf{w}} = (\mathbf{G}^H\mathbf{G} + \mu\mathbf{I})^{-1}\mathbf{G}^H\mathbf{G}\mathbf{y} \,. \tag{11}$$

The "clean signal" can be estimated as follows:

$$\hat{\mathbf{s}} = [\mathbf{I} - (\mathbf{G}^H\mathbf{G} + \mu\mathbf{I})^{-1}\mathbf{G}^H\mathbf{G}]\mathbf{y} \,. \tag{12}$$

At this stage some comments are in order. First, we note that when $\mu = 0$, $\hat{\mathbf{w}} = \mathbf{y}$. In other words, we have annihilated the signal ($\hat{\mathbf{s}} = 0$). If μ is too large, $\hat{\mathbf{w}} = \mathbf{0}$. In this case there is no snr enhancement ($\hat{\mathbf{s}} = \mathbf{y}$). In general, a line search procedure is used to determine the value of μ that yields a noise sequence with a mean square error that agrees with the estimated variance of the noise obtained after solving equation (9).

In Figure (1) we present a 1D synthetic example. The time sequence is composed of two real sinusoids immersed in noise with standard error $\sigma = 0.15$. The number of signals in this case is $p = 4$ (2 real sinusoids are represented by 4 complex harmonics). We have estimated the signal using different trade-off parameters μ. It is clear that for $\mu = 0.001$ we do not properly model the signal. When $\mu = 10$ our estimate of the signal is equal to the noisy signal. The optimum trade-off parameter is $\mu = 0.01$. This value yields a noise sequence with variance $\hat{\sigma} = 0.14$.

In Figure (2) we compare the performance of Canales' $f-x$ AR filter using a 10 points PEF and the $f-x$ ARMA approach using a 4 points PEF ($p = 3$). It is clear that the AR filter cannot properly separate signal from noise, this is a consequence of using an incorrect model (AR model) to design the PEF filter.

Projection Filters

Our approach is similar to the $f-x$ projection filter procedure proposed by Soubaras (1994, 1995). However, our final estimator of the noise sequence (equation (11)) is obtained from the ARMA structure rather than by invoking the concept of quasi-predictivility (Soubaras, 1995).

In fact, the operator $(\mathbf{G}^H\mathbf{G}+\mu\mathbf{I})^{-1}\mathbf{G}^H\mathbf{G}$ in equation (11) is a multinotch filter with an amplitude response given by:

$$\frac{|\mathcal{G}(k)|^2}{|\mathcal{G}(k)|^2 + \mu} \,.$$

In the last equation k denotes wavenumber and $\mathcal{G}$ stands for the Fourier transform of the PEF. Soubaras (1994) has indicated that this multnotch filter is equivalent to a projection operator that selectively filters the p wavenumbers associated to the deterministic part of the harmonic model.

Conclusion

We have shown that the correct representation of a superposition of linear events in the $f-x$ domain leads to an ARMA system. For nonlinear events, the data can be subdivided into smaller panels where the events are approximately linear. The signal can be modeled via a special type of ARMA process in which the autoregressive and the moving-average coefficients are identical.

The ARMA coefficients are computed after solving an eigenvalue problem. We have shown that the PEF is also the eigenvector associated to the minimum eigenvalue of the data covariance matrix. The noise attenuation process consists of two stages. First, we apply the PEF to the noisy data to estimate a colored noise sequence or filtered sequence. Finally, the PEF is deconvolved from the filtered sequence to estimate the additive noise.

It is clear that the second stage of our algorithm is equivalent to the projection filtering technique proposed by Soubaras (1994,1995). However, our equations are directly derived from the ARMA representation of the signal rather that by invoking the concept of quasi-predictivility. In Soubaras' technique the PEF is estimated from the data using an iterative process. In our approach the PEF is computed by solving an eigenvalue-eigenvector problem.

Finally, it is important to mention that the proposed model can be extended to a superposition of damped harmonics of the form $e^{-\alpha x + ikx}$. In this case, the covariance matrix of the problem corresponds to the non-Toeplitz or

Blackman-Tukey estimator. This estimator has been applied to the analysis of the Earth's free oscillations (Hori et al, 1989; Ulrych and Sacchi, 1995).

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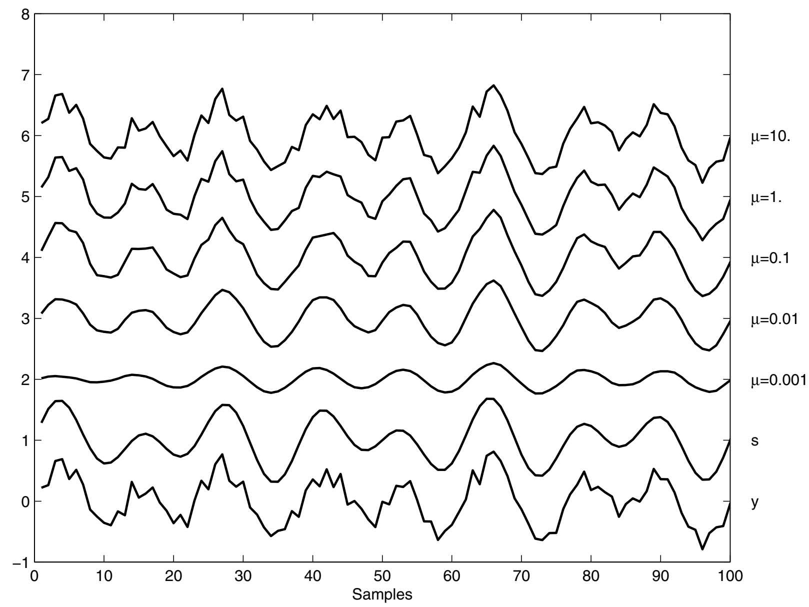

Fig. 1: The true signal s_k and the noisy signal y_k are displayed together with the estimated "clean signal" as a function of the tradeoff parameter μ (equation (12)). The optimum tradeoff parameter $\mu = 0.01$ yields a noise sequence with standard error $\hat{\sigma} = 0.14$. The standard error of the original noise sequence used to generated y_k is $\sigma = 0.15$.

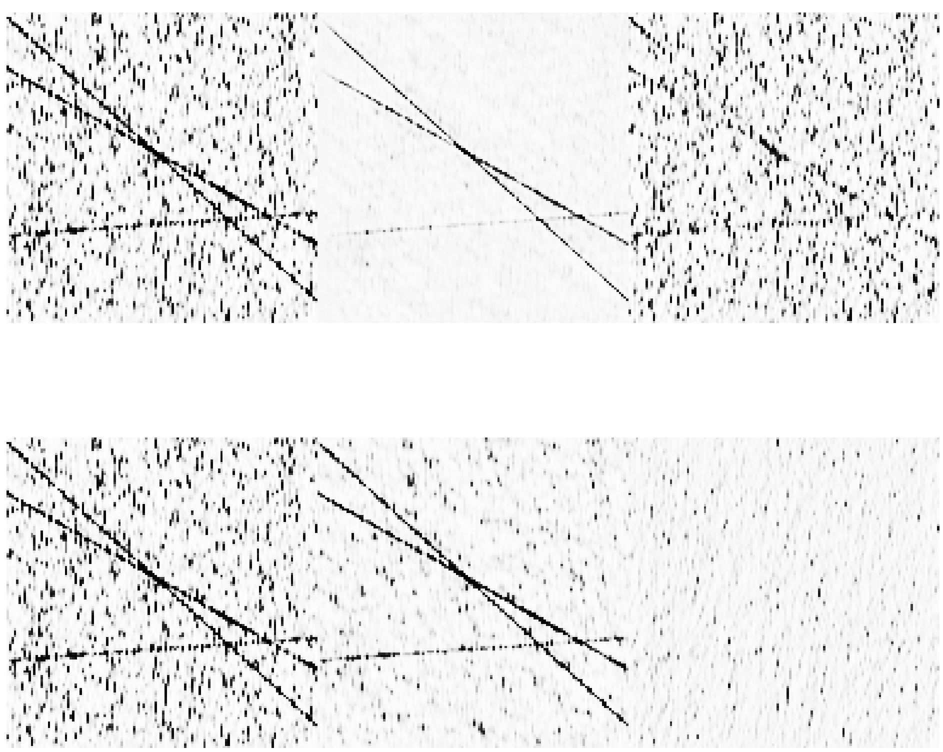

Fig. 2: Top: SNR enhancement using $f-x$ prediction error filters assuming an AR model (Canales, 1984). Bottom: SNR enhancement using an ARMA model. In both figures we portray the noisy data (left), the filtered data (center), and the residual panel (right).