Nonminimum-phase wavelet estimation using higher order statistics

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Seismic data are often represented by the well known convolutional model:

$$s(t) = w(t) \otimes r(t) + n(t) \quad (1)$$

where $w(t)$ denotes the seismic wavelet, $r(t)$ the reflectivity series, and $n(t)$ additive noise. The goal of seismic deconvolution is to design a filter $f(t)$ capable of removing or compressing the wavelet. To understand the effect of removing the wavelet from the seismogram, convolve both sides of equation (1) with the filter

$$f(t) \otimes s(t) = f(t) \otimes w(t) \otimes r(t) + f(t) \otimes n(t) \quad (2)$$

The last equation says that deconvolution can be successfully carried out if and only if two conditions are satisfied:

$$f(t) \otimes w(t) \approx \delta(t) \quad (3)$$
$$f(t) \otimes n(t) = e(t) \approx 0. \quad (4)$$

When these two conditions are simultaneously satisfied, one can write:

$$f(t) \otimes s(t) \approx r(t). \quad (5)$$

Equations (3) and (4) define a filter-design problem. However, in order to design $f(t)$, one must first know the wavelet, $w(t)$. Unfortunately, in most exploration scenarios, $w(t)$ is not known and, therefore, needs to be estimated.

This article focuses on estimating a wavelet with unknown amplitude and phase spectrum. Explicitly, the problem has one noisy observation (the seismogram) and two unknowns (the reflectivity and the seismic wavelet). We will examine how under certain conditions the stochastic nature of the reflectivity sequence can be exploited to estimate the wavelet.

Stochastic wavelet estimation. Stochastic wavelet estimation is often divided into two distinct problems:

Problem 1 (estimation): Under proper assumptions, we can compute a functional of the wavelet directly from $s(t)$, the observed data. The functional must be capable of reducing the unknown random $r(t)$ to a tractable quantity. It is also desirable to have a functional that annihilates the additive random noise $n(t)$. In mathematical terms, we seek an operator F such that

$$\mathrm{F}[s(t)] = k_r \bullet \mathrm{G}[w(t)] \quad (6)$$

where k_r is a constant and G another functional to determine.

Problem 2 (spectral factorization): This can be summarized as: Given an estimate of $\mathrm{G}[w(t)]$, how do we estimate $w(t)$? The latter leads to a new problem: Is $w(t)$ uniquely determined by $\mathrm{G}[w(t)]$?

The importance of non-Gaussianity. The idea of exploiting non-Gaussianity dates to Wiggins' MED (minimum entropy deconvolution) which assumes that the reflectivity is a sparse time series. In this context, sparseness can be associated with non-Gaussianity. In MED, an inverse operator is iteratively retrieved by maximizing a measure of sparseness. This is also a metric that measures departure from Gaussianity. The final estimator of the wavelet is computed by inverting the MED operator.

Our analysis assumes that the reflectivity is a white noise time series with either a Gaussian or non-Gaussian distribution and that the seismic trace can be represented by the linear system given in equation (1). The validity of equation (1) can be challenged. In fact, the reflectivity is not always a white-noise process, and the wavelet might not be time invariant. These effects can be quite important in real data problems but that is beyond the scope of this article.

The following three postulates, which we shall not try to prove, accentuate the importance of non-Gaussianity in nonminimum-phase wavelet estimation:

1) If $r(t)$ is Gaussian and $w(t)$ is a minimum-phase sequence, autocorrelation-based methods (second-order statistics) will correctly identify the amplitude and phase of the wavelet.
2) If $r(t)$ is Gaussian and $w(t)$ is nonminimum phase, no technique will correctly identify the phase of the wavelet.
3) If $r(t)$ is non-Gaussian and $w(t)$ is nonminimum phase, the amplitude and phase of the wavelet can be recovered if we know the actual distribution of $r(t)$.

Statement 3 clearly suggests that non-Gaussianity plays an important

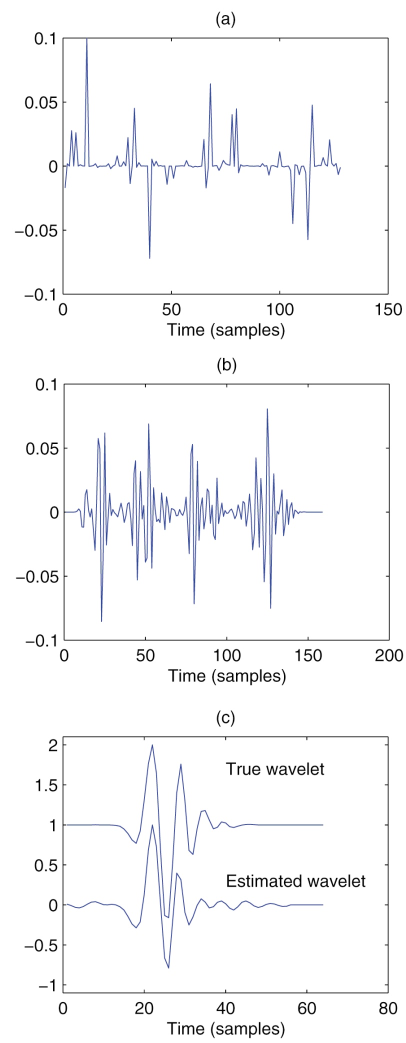

Figure 1. (a) A non-Gaussian reflectivity. (b) Seismic trace. (c) True wavelet and wavelet estimated by bispectral factorization.

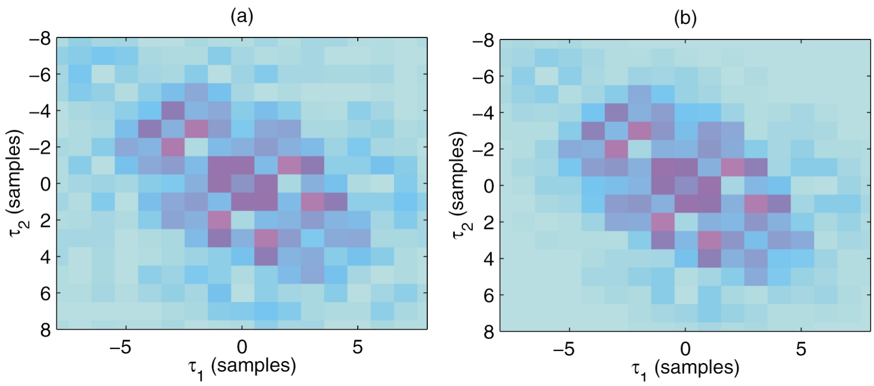

Figure 2. (a) Third-order cumulant of the data in Figure 1b. (b) The third-order moment of the true wavelet.

role in nonminimum-phase wavelet estimation. Higher-order spectra are defined in terms of higher-order cumulants and are useful to describe non-Gaussian processes. Quite contrary to power spectral density estimates (second-order statistics), higher-order spectra are capable of retaining phase information and, therefore, are useful for estimating nonminimum-phase wavelets.

In our approach to stochastic wavelet estimation, F is the n^{th} order cumulant of the data, and G is the n^{th} order moment of the wavelet (cumulants and moments are defined in the appendix). It is easy to show that, for a non-Gaussian white reflectivity, the n^{th} order cumulant of the seismic trace is given by

$$c_n^s(\tau_1, \tau_2, \dots \tau_{\{n-1\}}) = \gamma_n^r \sum_k w(k)w(k+\tau_1)\dots w(k+\tau_{\{n-1\}}), \quad (7)$$

where the variable γ_n^r is the central lag of the n^{th} order cumulant of the reflectivity (the other lags having vanished). The term lag implies, essentially, the time shift of the time series with respect to itself.

The term on the right is also the n^{th} order moment of the seismic wavelet multiplied by the scalar γ_n^r. We will examine only the cases when $n = 2,3,4$. The second-order cumulant $c_2^s(\tau)$ is also called the autocorrelation function of the trace. In this case, γ_n^r is the variance of the reflectivity series, and the term under the summation symbol is the second-order moment of the wavelet. For $n = 3$ (third-order statistics), $c_3^s(\tau_1,\tau_2)$ is the third-order cumulant, γ_3^r is the skewness of the reflectivity, and the term under the summation symbol is the third-order moment of the data. Similarly, the fourth-order cumulant is obtained for $n = 4$; in this case, γ_4^r is the kurtosis of the reflectivity (variance, skewness, and kurtosis are defined both mathematically and semantically in the appendix).

At this point some comments are in order. For Gaussian processes, the higher-order cumulants $n = 3,4\dots$ vanish for all lags, i.e., $\gamma_n^r = 0$ for $n = 3,4$ ... On the other hand, if the process is Gaussian, the central lag of the second-order cumulant does not vanish ($\gamma_2^r \neq 0$). As already mentioned, the latter is the variance of the white noise reflectivity.

The z-transform of the n^{th} order cumulant of the wavelet is

$$C_n^s(z_1, z_2, \dots z_{\{n-1\}}) = \gamma_n^r W(z_1)W(z_2)\dots W(z_1^{-1} z_2^{-1} \dots z_{\{n-1\}}^{-1}) \quad (8)$$

If this equation is evaluated on the unit circle ($|z|=1$), we obtain the expressions for the amplitude spectrum, $n = 2$, and for higher-order spectra $n = 3,4\dots$, respectively.

Let us consider the second-order statistics case, $n = 2$,

$$C_2^s(z) = \gamma_2^r W(z)W(z^{-1}) = \gamma_2^r |W(z)|^2 \quad (9)$$

In this case, the amplitude spectrum of the trace is equal to the amplitude spectrum of the wavelet multiplied by a scale factor. It is important to stress that the wavelet phase is lost. The phase needs to be imposed via an extra assumption. In general, the wavelet is assumed to be a minimum-phase signal. It is interesting to note that for higher-order cumulants the phase of the wavelet is retained, and therefore it can be estimated. The expressions for the higher spectra obtained from the

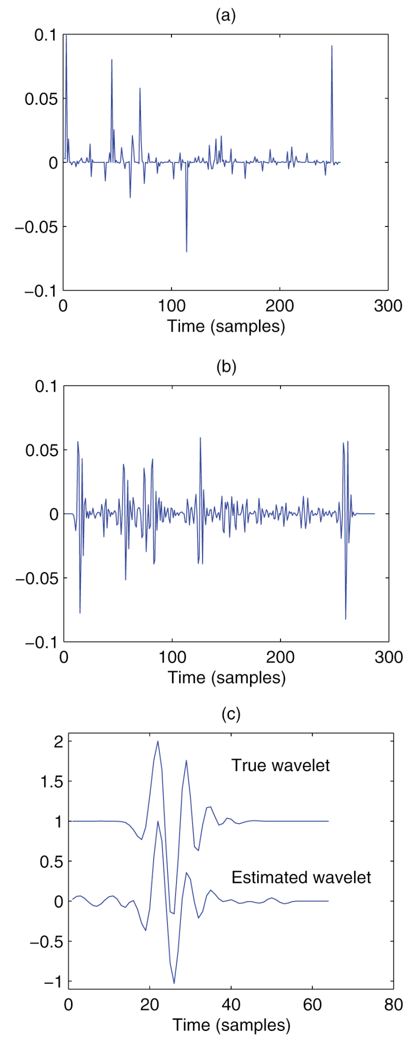

Figure 3. (A) A non-Gaussian reflectivity. (b) Seismic trace. (c) True wavelet and wavelet estimated by trispectral factorization.

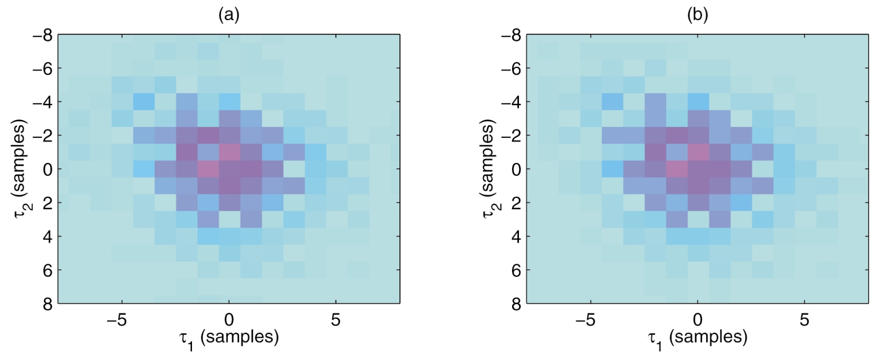

Figure 4. (a) A slice of the fourth-order cumulant of the data in Figure 3a. (b) A slice of the fourth-order moment of the true wavelet.

third and fourth-order cumulants are given by:

$$C_3^s(z_1,z_2) = \gamma_3^r\, W(z_1)W(z_2)W(z_1^{-1}z_2^{-1}) \qquad (10)$$

$$C_4^s(z_1,z_2,z_3) = \gamma_4^r\, W(z_1)W(z_2)W(z_3)W(z_1^{-1}z_2^{-1}z_3^{-1}) \qquad (11)$$

The spectrum of the third-order cumulant is the bispectrum [equation (10)], and the spectrum of the fourth-order cumulant is the trispectrum. As an exercise, one can substitute into the previous expressions $W(z) = |W(z)|\,e^{i\phi(z)}$ and prove that the phase information is not annihilated.

A few words about the differences between third- and fourth-order statistics are needed. The skewness is zero except for asymmetrically distributed processes. This might not be a good assumption on which to model seismic reflectivity. In short windows, one might expect a bias in the amplitude of the reflectors and, therefore, the skewness should not vanish. On the other hand, the fourth-order cumulant preserves phase information for both non-Gaussian symmetrically and nonsymmetrically distributed reflectivities. This is why we often prefer the fourth-order cumulant over the third-order for wavelet estimation.

Spectral factorization. Many time series have a given autocorrelation function. Spectral factorization, in the context of second-order statistics, is done to find one time series that is also minimum phase. When higher-order statistics is invoked, the minimum phase assumption is not required. In our approach to wavelet estimation, the spectral factorization for second- and higher-order statistics uses the Kolmogoroff technique (see Claerbout, 1985). In this method, we take natural logarithm of the n^{th} order cumulant

$$\ln\,[C_n^s(z_1,z_2,\ldots z_{\{n\text{-}1\}}] = \ln[\gamma_n^r] + \ln\,[W(z_1)] + \ln[W(z_2)]\ldots + \ln\,[W(z_1^{-1}z_2^{-1}\ldots z_{\{n\text{-}1\}}^{-1}] \qquad (12)$$

By expanding the logarithm in a Laurent series, it is easy to obtain an expression that allows us to compute the minimum-phase wavelet when $n = 2$ or a mixed-phase wavelet when $n = 3,4$. For the higher-order case, the minimum- and maximum-phase components of the wavelet can be estimated separately. In the numerical algorithm, the z-transform is evaluated on the unit circle using FFTs (see Sacchi et al, 1998).

We illustrate the wavelet-estimation problem with synthetic examples. We first estimate the wavelet from a non-Gaussian process with nonzero skewness using the third-order cumulant. In this case, the seismic trace was simulated by convolving a wavelet with a sparse reflectivity generated by randomly setting to zero samples from a Gaussian distribution. A different probability was given according to the sign of the sample. This is to guarantee a non-Gaussian process with nonzero skewness. This example used 128 points to estimate the third-order cumulant. Figures 1a and 1b show the seismic reflectivity and the seismic trace. The resulting wavelet is Figure 1c. Figure 2a shows the third-order cumulant of the trace. Figure 2b is an estimate of the third-order moment of the wavelet.

A similar analysis was carried out with the fourth-order cumulant (Figures 3 and 4). In this case, a time series of 256 points was used. In general, estimation of the fourth-order cumulant from the data is not easy. The procedure is computationally intensive and also requires long data records. This requirement cannot always be satisfied when working with real data.

Cumulant matching. Another way to estimate the wavelet is by cumulant matching, i.e., finding a wavelet that matches the cumulant of the trace. The problem entails optimization of a nonlinear misfit function that can be minimized by linearized inversion (Lazear, 1993) or global optimization (Velis and Ulrych, 1996).

A hybrid strategy that combines second- and higher-order statistics is also plausible. The autocorrelation of the trace can be used to retrieve a minimum-phase wavelet. The roots of the minimum-phase wavelet, which lie outside the unit circle, can be mapped inside the unit circle to explore the space of plausible wavelets with the same autocorrelation function. A cumulant-matching criterion based on third- or fourth-order statistics is then used to estimate the nonminimum-phase wavelet that honors the second-order constraint (the autocorrelation) and the higher-order constraints (the third- or fourth-order cumulant). This technique has been applied to real data with encouraging results in the case of wavelets with well defined maximum-phase components (see Sacchi, 1999).

Conclusions. Second-order statistics answer the problem of wavelet estimation when the reflectivity is white and the wavelet is considered minimum phase. By imposing an extra assumption, non-Gaussian reflectivity, higher-order statistics can be used to extract nonminimum-phase wavelets. The problem of estimating a nonminimum-phase wavelet from a higher-order cumulant is analogous to the problem of estimating a minimum-phase wavelet from the autocorrelation. In particular, the celebrated Kolmogoroff spectral factorization, which is implemented via the Hilbert transform, can be generalized to the factorization of higher-order spectra.

Suggestions for further reading. *Fundamentals of Geophysical Data Processing* by Claerbout (Blackwell, 1985). "Mixed-phase wavelet estimation using fourth-order cumulants" by Lazear (GEOPHYSICS, 1993). "Non-casual non-minimum phase ARMA modeling of non-Gaussian processes" by Petropulu (IEEE *Transactions on Signal Processing*, 1995). "Non-minimum phase wavelet estimation using polycepstra" by Sacchi et al. (*Journal of Seismic Exploration*, 1998). "A procedure for wavelet estimation based on second and fourth order statistics" by Sacchi (1999 CSEG *Annual Meeting Abstracts*). "Simulated annealing wavelet estimation via fourth-order cumulant matching" by Velis and Ulrych (GEOPHYSICS, 1996). E

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Appendix. Sample estimates of second-order and higher-order cumulants can be estimated as follows:

$$c_2^s(\tau) = \{1/N\}\sum_n s(n)s(n+\tau),$$

$$c_3^s(\tau_1,\tau_2) = \{1/N\}\sum_n s(n)s(n+\tau_1)s(n+\tau_2),$$

$$c_4^s(\tau_1,\tau_2,\tau_3) = \{1/N\}\sum_n s(n)s(n+\tau_1)s(n+\tau_2)s(n+\tau_3) - c_2^s(\tau_1)c_2^s(\tau_3-\tau_2) - c_2^s(\tau 2)c_2^s(\tau_3-\tau^1) - c_2^s(\tau_3)c_2^s(\tau_2-\tau_1)$$

For a zero mean process, the second- and third-order cumulants are equivalent to second- and third-order moments, respectively. This is not the case for the fourth-order moment which is defined as

$$m_4^s(\tau_1,\tau_2,\tau_3) = \{1/N\}\sum_n s(n)s(n+\tau_1)s(n+\tau_2)s(n+\tau_3).$$

The zero lag coefficient of the second-order cumulant, $c_2^s(0)$, is the variance of the time series. The variance is a measure of the energy in a time series.

The zero lag coefficient of the third-order cumulant, $c_3^s(0,0)$, is an estimate of the skewness. This is a measure of the asymmetry of the associated probability distribution. Similarly, an estimate of the kurtosis is given by $c_4^s(0,0,0)$. The kurtosis of a Gaussian distribution is zero. The kurtosis is a measure of the contribution of the tails of a distribution to the total area under the distribution curve.